Pseudofinity of Vertex-transitive and Edge-transitive graphs

Nursultan Bekbossyn,* Yerassyl Altynbek† and under direction of Nurlan Markhabatov
Department of Mechanics and Mathematics, Eurasian National University

(August 15R)

This research focuses on proving the pseudonofinity of (n, k)-star graphs, a class of interconnection networks. We aim to establish the conditions under which these graphs exhibit infinite behavior, contributing to the understanding of their scalability and robustness in network topologies. The results will have potential applications in parallel computing and communication systems.

PACS numbers:
Keywords: (n, k)-star graphs Pseudonofinity Interconnection networks Graph theory Scalability Network topologies Parallel computing Communication systems Combinatorial properties Infinite graph sequences Approxmination theory

I. INTRODUCTION

Recently, pseudofinite graphs have been actively studied by various authors in various fields. In [17], graphs interpreted in pseudofinite fields were studied. The papers [18, 19] introduce various ultraflat Hrushovski-Fraïssé classes of acyclic graphs whose limits are strictly superstable and pseudofinite. In [12], generic pseudofinite graph theories are noted. Tao's work [17] concerns graphs definable in (pseudo)finite fields.

In the works [5,6,15] (super)stable graphs and their important properties were studied. The article [8] is devoted to the description of types of pregeometries with an algebraic closure operator for acyclic graph theories.

In [11], the following theorem was proved

Theorem 1. Any theory T of a regular graph with an infinite model is pseudofinite.

This theorem immediately implies

Theorem 2. [3] A theory T_r is the theory of an infinite tree, each vertex of which has degree r. The theory T_∞ (also known as the everywhere infinite forest theory) is the theory of an infinite tree in which each vertex has infinite degree. Both theories and are pseudofinite.

Note also that the theory T_∞ is the Fraïssé limit of finite trees.

Let us recall the basic definitions of approximations of the theory and the construction of the so-called "coproduct". And all other terminalologies from graph theory and model theory can be found in [4,9,10].

Recall the following basic concepts. Let $L = \{R\}$, where R is a binary relation symbol. A *graph* is a structure $\Gamma = \langle G, R\rangle$. The elements of the universe G are called the *vertices* of the graph Γ, and the elements of the binary relation $R \subseteq G^2$ are *edges*. The edge connecting the vertices a and b is denoted by $[a, b]$. If a vertex $a \in G$ is an endpoint of edge $u \in R$, then a and u are *incident*. The *degree* of vertex a in a graph Γ, written $\deg_\Gamma(a)$ or simply $\deg(a)$, is the number of edges incident to a, except that each loop at a counts twice. A vertex of degree 0 is called *isolated*, a vertex of degree 1 is called *leaf* or *hanging vertex*. A graph that contains no cycles is called an *acyclic graph*. A connected acyclic graph is called a *tree*. Any graph without cycles is also called a *forest* so that the components of a forest are trees. Subsystems $\Gamma' = \langle G', R'\rangle$ of the graph Γ are called *subgraphs*. If $\Gamma' \subseteq \Gamma$ and Γ' contains all the edges $[x, y] \in R'$ with $x, y \in G'$, then Γ' is an *induced subgraph* of Γ.

Theorem 3. *A*ny nontrivial finite tree has at least two hanging vertices.

*E-mail: nursultan.bekbossyn@astanaboysbil.edu.kz
†E-mail: yerassylaltynbek@gmail.com

Definition 1. An (n,k)-star graph, denoted by $S_{n,k}$, is specified by two integers n and k, where $1 \leq k < n$. The vertex set of $S_{n,k}$ is represented as $\{p_1p_2 \dots p_k \mid p_i \in \langle n \rangle \text{ and } p_i \neq p_j \text{ for } i \neq j\}$. Adjacency in $S_{n,k}$ is defined as follows:

1. Two vertices $p_1p_2 \dots p_i \dots p_k$ and $p_ip_2 \dots p_1 \dots p_k$ are adjacent through an edge of dimension i, where $2 \leq i \leq k$. This involves swapping p_1 with p_i. These edges are called i-edges.
2. Two vertices $p_1p_2 \dots p_k$ and $xp_2 \dots p_k$ are adjacent through an edge of dimension 1, where $x \in \langle n \rangle - \{p_i \mid 1 \leq i \leq k\}$. These edges are referred to as 1-edges.

A $(4,2)$-star graph is shown in Figure 1.

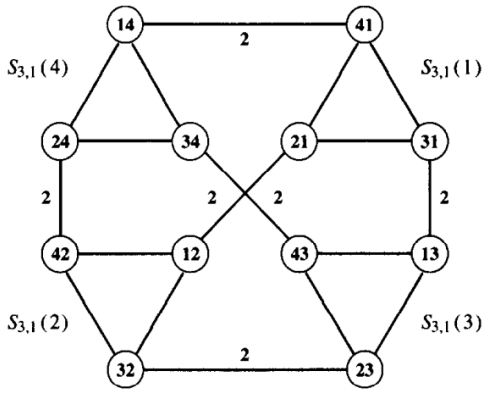


Fig. 1. A (4,2)-star graph.

1. Background to pseudofinite theories

In the work of J. Ax [1] the concept of pseudofiniteness was first defined. The ground works obtained to date for pseudofinite structures directly depend on the results of J. Ax. The basic definitions of pseudofiniteness are as follows.

Definition. [1] An infinite structure $\mathcal{M}$ of a fixed language L is *pseudofinite* if for all L-sentences φ, $\mathcal{M} \models \varphi$ implies that there is a finite $\mathcal{M}_0$ such that $\mathcal{M}_0 \models \varphi$. The theory $T = Th(\mathcal{M})$ of the pseudofinite structure $\mathcal{M}$ is called *pseudofinite*.

Many beautiful theorems in model theory of the 1950-60s were proved using the ultraproducts. Set theorists prefer to use ultraproducts since they give rise to elementary embeddings, a staple of large cardinal theory. J. Ax in [1] connect the notion of pseudofiniteness and the construction of ultraproducts.

Proposition 1.2. [1] *Fix a language L and an L-structure $\mathcal{M}$. Then the following are equivalent:*
1) an L-structure $\mathcal{M}$ is pseudofinite;
2) $\mathcal{M} \models T_f$, where T_f is the common theory of all finite L-structures.;
3) $\mathcal{M}$ is elementarily equivalent to an ultraproduct of finite L-structures.

In classical logic, the following property is a straightforward consequence of pseudofiniteness.

Proposition 1.3. *Let $\mathcal{M}$ be a pseudofinite structure and $f : M^k \to M^k$ be a definable function. Then f is injective if and only if f is surjective.*

2. Background to approximation theories

Definition. [**?**] Let $\mathcal{T}$ be a family of theories and T be a theory such that $T \notin \mathcal{T}$. The theory T is said to be $\mathcal{T}$*-approximated,* or *approximated by the family* $\mathcal{T}$, or a *pseudo-*$\mathcal{T}$*-theory,* if for any formula $\varphi \in T$ there exists $T' \in \mathcal{T}$ for which $\varphi \in T'$.

If the theory T is $\mathcal{T}$-approximated, then $\mathcal{T}$ is said to be an *approximating family* for T, and theories $T' \in \mathcal{T}$ are said to be *approximations* for T. We put $\mathcal{T}_\varphi = \{T \in \mathcal{T} \mid \varphi \in T\}$. Any set $\mathcal{T}_\varphi$ is called the φ*-neighbourhood*, or simply a *neighbourhood*, for $\mathcal{T}$. An approximating family $\mathcal{T}$ is called *e-minimal* if for any sentence $\varphi \in \Sigma(\mathcal{T})$, $\mathcal{T}_\varphi$ is finite or $\mathcal{T}_{\neg\varphi}$ is finite. It was shown in [**?**] that any *e-minimal* family $\mathcal{T}$ has unique accumulation point T with respect to neighbourhoods $\mathcal{T}_\varphi$, and $\mathcal{T} \cup \{T\}$ is also called *e-minimal*.

Recall that the *E-closure* for the family $\mathcal{T}$ of complete theories is characterized by the following proposition.

Proposition 1.4. [2] *Let $\mathcal{T}$ be a family of complete theories of the language Σ. Then $Cl_E(\mathcal{T}) = \mathcal{T}$ for a finite $\mathcal{T}$, and for an infinite $\mathcal{T}$, a theory T belongs to $Cl_E(\mathcal{T})$ if and only if T is a complete theory of the language Σ and $T \in \mathcal{T}$, or $T \neq \mathcal{T}$ and for any formula φ the set $\mathcal{T}_\varphi$ is infinite.*

We denote by $\overline{\mathcal{T}}$ the class of all complete elementary theories of relational languages, by $\overline{\mathcal{T}}_{fin}$ the subclass of $\overline{\mathcal{T}}$consisting of all theories with finite models, and by $\overline{\mathcal{T}}_{inf}$ the class $\overline{\mathcal{T}} \backslash \overline{\mathcal{T}}_{fin}$.

Proposition 1.5. [**?**] *For any theory T the following conditions are equivalent:*

(1) T is pseudofinite;

(2) T is $\overline{\mathcal{T}}_{fin}$-approximated;

(3) $T \in Cl_E(\overline{\mathcal{T}}_{fin}) \backslash \overline{\mathcal{T}}_{fin}$.

Definition 1. [20] *A disjoint union* $\bigsqcup_{n\in\omega} \mathcal{M}_n$ of pairwise disjoint systems $\mathcal{M}_n$ of pairwise disjoint predicate signatures $\Sigma_n, n \in \omega$, is a system of signature $\bigcup_{n\in\omega} \Sigma_n \cup \{P_n^{(1)} | n \in \omega\}$ with support $\bigsqcup_{n\in\omega} M_n$, $P_n = M_n$, and interpretations of predicate symbols from Σ_n that coincide with their interpretations in systems $\mathcal{M}_n$, $n \in \omega$.

A disjoint union of theories T_n, pairwise disjoint predicate signatures Σ_n, respectively, $n \in \omega$, is the theory

$$\bigsqcup_{n\in\omega} T_n \rightleftharpoons Th(\bigsqcup_{n\in\omega} \mathcal{M}_n),$$

where $\mathcal{M}_n \models T_n, n \in \omega$.

Obviously, the $T_1 \sqcup T_2$ theory does not depend on the choice of the disjunctive union $\mathcal{M}_1 \sqcup \mathcal{M}_2$ of the models $\mathcal{M}_1 \models T_1$ and $\mathcal{M}_2 \models T_2$.

In [**?**] it was shown that in the study of uncountable categoricity and ω-stability a special type of structures is of principal importance.

Definition A structure $\mathcal{M}$ is said to be *minimal* if any subset definable in the structure using parameters is either finite or co-finite (a complement to a finite set). $\mathcal{M}$ is said to be *strongly minimal* if any model of $Th(\mathcal{M})$ is minimal.

The notion of strong minimality is important in model theory because it provides a way to classify theories based on the complexity of their definable sets. Strongly minimal theories have many interesting and useful properties, including simplicity and stability, which make them amenable to study and applications in other areas of mathematics.

In graph theory, this concept can be adapted to characterize minimal graph structures where any definable subset is either small (finite) or almost complete (co-finite).

II. CONCLUSIONS

ACKNOWLEDGMENTS

REFERENCES

[1] Ax, J. (1968). The Elementary Theory of Finite Fields. *Annals of Mathematics, 88*(2), 239–271. https://doi.org/10.2307/1970573

[2] Sudoplatov, S.V. (2016). Closures and generating sets related to combinations of structures. *The Bulletin of Irkutsk State University. Series Mathematics, 16*, 131–144.

[3] Garcia D., Robles M. Pseudofiniteness and measurability of the everywhere infinite forest. (2023) https://doi.org/10.48550/arXiv.2309.00991

[4] Grinberg, D. An introduction to graph theory. 2023. arXiv preprint arXiv:2308.04512. https://doi.org/10.48550/arXiv.2308.04512
[5] Herre, Heinrich, Mekler, Allan, and Smith, Kenneth. "Superstable graphs." Fundamenta Mathematicae 118.2 (1983): 75-79.
[6] Ivanov, A.. The structure of superflat graphs. Fundamenta Mathematicae 143.2 (1993): 107-117.
[7] Kantor W.M., Liebeck M.W., Macpherson H.D. $\aleph_0$-categorical structures smoothly approximated by finite substructures, *Proc. London Math. Soc.* 1989. vol. 59. pp. 439–463. https://doi.org/10.1112/plms/s3-59.3.439
[8] Malyshev S. B. Kinds of Pregeometries of Acyclic Theories. The Bulletin of Irkutsk State University. Series Mathematics, 2023, vol. 46, pp. 110–120. https://doi.org/10.26516/1997-7670.2023.46.110
[9] Marker D. *Model Theory: An Introduction.* Graduate Texts in Mathematics, vol. 271. Springer Verlag, New York, Berlin and Heidelberg, 2002, 342 p.
[10] Markhabatov N. D. Approximations of Acyclic Graphs. *The Bulletin of Irkutsk StateUniversity. Series Mathematics*, 2022, vol. 40, pp. 104–111. https://doi.org/10.26516/1997-7670.2022.40.104
[11] Markhabatov N. D., Sudoplatov S. V., Approximations of Regular Graphs, *Herald of the Kazakh-British Technical University*, 2022, vol.19, no. 1, pp. 44–49 https://doi.org/10.55452/1998-6688-2022-19-1-44-49
[12] Myasnikov, A.G., Remeslennikov, V.N. Generic Theories as a Method for Approximating Elementary Theories. *Algebra and Logic* 2015, vol. 53, pp. 512–519 https://doi.org/10.1007/S10469-015-9314-0
[13] Nurtazin A.T. Graphs and Models with finite chains, *Siberian Electronic Mathematical Reports,* 2007, volume 4, pp. 238–248
[14] Ovchinnikova E.V., Shishmarev Yu.E., (1988). Countably categorical graphs, *Ninth All-Union Conference on Mathematical Logic. Leningrad, September 27-29, dedicated to the 85th anniversary of Corresponding Member of the USSR Academy of Sciences A.A. Markov*: abstracts, p.120
[15] Podewski, Klaus-Peter, and Ziegler, Martin. "Stable graphs." Fundamenta Mathematicae 100.2 (1978): 101-107.
[16] Sudoplatov S. V., Approximations of theories, *Siberian Electronic Mathematical Reports*, 2020, vol.17, pp. 715–725
[17] Tao T. Expanding polynomials over finite fields of large characteristic, and a regularity lemma for definable sets. *Contributions to Discrete Mathematics*. Volume 10, Number 1, Pages 22–98, ISSN 1715-0868. 2014
[18] Valizadeh, A.N., Pourmahdian, M. Pseudofiniteness in Hrushovski Constructions. *Notre Dame J. Formal Log.* 2020, vol. 61, pp. 1–10. https://doi.org/10.1215/00294527-2019-0038
[19] Valizadeh, A.N., Pourmahdian, M. Strict Superstablity and Decidability of Certain Generic Graphs. *Bull. Iran. Math. Soc.* 2019, vol. 45, pp. 1839–1854. https://doi.org/10.1007/s41980-019-00234-2
[20] Woodrow R.E. *Theories with a finite number of countable models and a small language. Ph. D. Thesis.* Simon Fraser University, 1976, 99 p.